Fourier Analysis

April 14, 2022

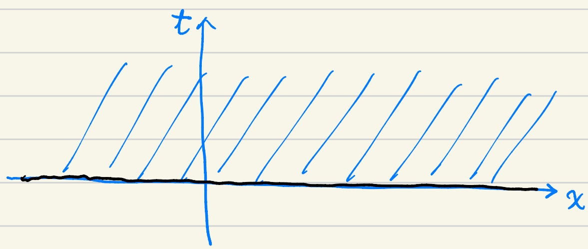
Review.

Thm (uniqueness).

Suppose $u = u(x,t)$ satisfies the following conditions:

① $u \in C(\overline{\mathbb{R}\times\mathbb{R}_+}) \cap C^2(\mathbb{R}\times\mathbb{R}_+)$,

where $\overline{\mathbb{R}\times\mathbb{R}_+} = (-\infty,\infty)\times[0,\infty)$.



② $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ on $\mathbb{R}\times\mathbb{R}_+$

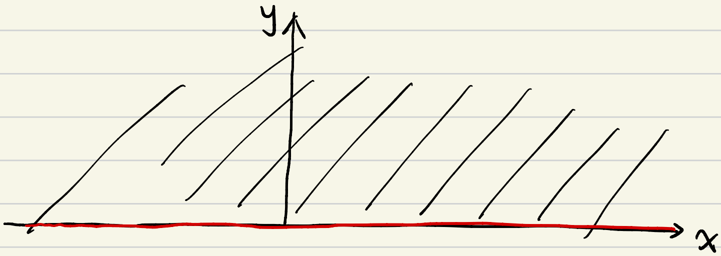
③ $u(x,0) = 0, \quad x \in \mathbb{R}$

④ $u(\cdot, t)$ belongs to $S(\mathbb{R})$ uniformly in t.

Then $u(x,t) \equiv 0$ on $\mathbb{R}\times\mathbb{R}_+$

• The proof applies an energy method.

Application 2: Steady state heat equation on the upper half plane.



$u = u(x,y)$ temperation distribution

$$\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0. & (1) \\ u(x,0) = f(x). & (2) \end{cases}$$

Now we use Fourier transform to derive a solution by some formal arguments.

Taking Fourier transform in x-variable in ① gives

$$(2\pi i \xi)^2 \hat{u}(\xi, y) + \frac{\partial^2 \hat{u}(\xi, y)}{\partial y^2} = 0$$

That is,
$$\frac{\partial^2 \hat{u}(\xi, y)}{\partial y^2} - 4\pi^2 \xi^2 \hat{u}(\xi, y) = 0$$

For fixed ξ, the above is a linear 2nd order ODE.

The general solution is

$$\hat{u}(\xi, y) = A(\xi)\, e^{-2\pi|\xi| y} + B(\xi)\, e^{2\pi|\xi|\cdot y}$$

We remove the second part since it is rapidly increasing.

Therefore

$$\hat{u}(\xi, y) = A(\xi)\, e^{-2\pi|\xi| y}.$$

Taking $y=0$,

$$\hat{u}(\xi, 0) = A(\xi) = \hat{f}(\xi)$$

That is

$$\hat{u}(\xi, y) = \hat{f}(\xi) \cdot e^{-2\pi|\xi| y}.$$

Now we introduce the Poisson kernel on the upper half plane:

$$P_y(x) = \frac{1}{\pi} \cdot \frac{y}{x^2+y^2}, \qquad x \in \mathbb{R}, \quad y > 0$$

Claim: $\widehat{P_y}(\xi) = e^{-2\pi|\xi|y}$ for $y>0$.

Hence $\hat{u}(\xi, y) = \hat{f}(\xi)\cdot\widehat{P_y}(\xi)$

$= \widehat{f * P_y}(\xi)$.

By Inversion formula,

$u(x,y) = f * P_y(x), \quad x\in\mathbb{R}, \quad y>0.$

Lem 1.

(1) $\int_{-\infty}^{\infty} e^{-2\pi|x|y} e^{-2\pi i\xi x} dx$

$= P_y(\xi)$

(2) $\int_{-\infty}^{\infty} P_y(\xi) e^{-2\pi i\xi x} d\xi = e^{-2\pi|x|y}$

Pf. Remember for $a>0$,

$$e^{-a|x|} \xrightarrow{\mathcal{F}} \frac{2a}{a^2+4\pi^2\xi^2}$$

Letting $a = 2\pi y$ gives

$$e^{-2\pi y|x|} \xrightarrow{\mathcal{F}} \frac{2\cdot 2\pi y}{(2\pi y)^2+4\pi^2\xi^2} = \frac{1}{\pi}\cdot\frac{y}{y^2+\xi^2}$$
$$= \mathcal{P}_y(\xi).$$

This proves ①.

By ① and Inversion formula,

$$\int \mathcal{P}_y(\xi)\, e^{+2\pi i\xi x}\, d\xi = e^{-2\pi y|x|}$$

Taking complex conjugate on both sides gives ②

Next we prove the following

Thm 2. Let $f \in \mathcal{S}(\mathbb{R})$. Let

$$u(x,y) = f * P_y(x), \quad x \in \mathbb{R}, \ y > 0.$$

Then u satisfies the following:

① $u \in C^2(\mathbb{R} \times \mathbb{R}_+)$ and $\Delta u = 0$

② $u(x,y) \rightrightarrows f(x)$ as $y \to 0$

③ $\int_{-\infty}^{\infty} |u(x,y) - f(x)|^2 \, dx \to 0$ as $y \to 0$

④ $u(x,y) \to 0$ as $|x| + y \to \infty$.

("u vanishes at ∞")

Pf. Here we only prove ④.

We will show that $\exists$ $C > 0$ such that

$$|u(x,y)| \leq \begin{cases} C \cdot \left(\dfrac{1}{1+x^2} + \dfrac{y}{x^2+y^2} \right), \\ \dfrac{C}{y} \end{cases}.$$

for all $x \in \mathbb{R}$, $y > 0$.

To this end, recall that

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2+y^2} \leq \frac{1}{\pi} \cdot \frac{1}{y}$$

Hence

$$|f * P_y(x)| \leq \int_{-\infty}^{\infty} |f(x-t)| \, |P_y(t)| \, dt$$

$$\leq \int_{-\infty}^{\infty} |f(x-t)| \cdot \frac{1}{\pi} \cdot \frac{1}{y} \, dt$$

$$= \frac{1}{\pi y} \cdot \int_{-\infty}^{\infty} |f(x)| \, dx$$

$$\leq \frac{C}{y},$$

To see the other part.

Notice that

$$f * P_y(x) = \int_{-\infty}^{\infty} f(x-t) \, P_y(t) \, dt$$

$$= \int_{|t| \leq \frac{|x|}{2}} + \int_{|t| > \frac{|x|}{2}} f(x-t) \, P_y(t) \, dt$$

$$= (\mathrm{I}) + (\mathrm{II})$$

Then

$$|(\mathrm{I})| \leq \int_{|t| \leq \frac{|x|}{2}} |f(x-t)| \; P_y(t)\, dt$$

$$\left(|x-t| \geq \frac{|x|}{2} \right)$$

$$\leq \int_{|t| \leq \frac{|x|}{2}} \frac{C}{1+\left(\frac{|x|}{2}\right)^2} \; P_y(t)\, dt$$

$$\leq \frac{C}{1+\left(\frac{|x|}{2}\right)^2} \cdot \underbrace{\int_{-\infty}^{\infty} P_y(t)\, dt}_{=1}$$

$$\leq \frac{4C}{1+|x|^2}.$$

$$|(\mathrm{II})| \leq \int_{|t| \geq \frac{|x|}{2}} |f(x-t)| \cdot \frac{1}{\pi} \; \frac{y}{y^2+t^2}\, dt$$

$$\leq \int_{|t| \geq \frac{|x|}{2}} |f(x-t)| \cdot \frac{1}{\pi} \; \frac{y}{y^2+\frac{|x|^2}{4}}\, dt$$

$$\leq \frac{4y}{\pi(y^2+4x^2)} \cdot \int_{-\infty}^{\infty} |f(t)|\, dt$$

$$\leq \quad \frac{\tilde{C}\, y}{y^2+x^2}.$$

$\blacksquare$